Calculus 2 Notebook

FINAL

منى سكجها

No.

* Alternating Test :-

Alternating series is of the form :-

$$\sum_{1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 + \cdots$$

<u>or</u>

$$\sum_{1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4$$

Alternating tests Given $\sum (-1)^{\overset{n-1}{\overset{n+1}{n}}} a_n$

if ① a_n is decreasing seq (i.e $a_n' < 0$).

and ② $\lim_{n\to\infty} a_n = 0$

then $\sum (-1) a_n$ is convergent

Note :- when you see Alternating series first check
① divergence test
② Alternating test
③ Rotating Root

④ convergence test

Ex: Is the following conv. or div

① $\sum (-1)^n \frac{2n+1}{3n+1}$

using div. test

$\lim_{n\to\infty} (-1)^n \cdot \frac{2n+1}{3n+1} = \pm 1 \cdot \frac{2}{3} = \pm \frac{2}{3} \neq 0$ div

② $\sum (-1)^n \cdot \frac{1}{n}$

using Alternating test $a_n = \frac{1}{n}$, $a_n' = \frac{-1}{n^2} < 0 \downarrow$

$\lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow$ conv

$$\boxed{\begin{array}{l} \sum \frac{1}{n} \text{ div} \\ \sum (-1)^n \cdot \frac{1}{n} \text{ conv} \end{array}}$$

③ $\sum_{3}^{\infty} (-1)^n \cdot \frac{\ln n}{n}$

using Alternating test $a_n = \frac{\ln n}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} = 0$ ✓

$a_n' = \frac{n \cdot \frac{1}{n} - \ln n \cdot 1}{n^2} < 0 \downarrow$

$\sum (-1)^n \frac{\ln n}{n}$ conv.

④ $\sum (-1)^n \frac{n}{\ln n}$

divergence test

$\lim_{n\to\infty} (-1)^n \cdot \frac{n}{\ln n}$

$\pm 1 \cdot \lim_{n\to\infty} \frac{n}{\ln n}$ L' rule

$\pm 1 \cdot \infty = \pm\infty \neq 0$ div

* convergence test

if $\sum |a_n|$ conv. $\Rightarrow \sum a_n$ conv.

$\sum a_n$

① $\frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots$

taking $|\,|$ $\Rightarrow \sum |a_n| \Rightarrow \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ geom. $|r| = \frac{1}{2} < 1$ conv.

using conv. test $\sum a_n$ conv.

② $\sum_{1}^{\infty} \frac{\cos n}{n^2}$ using convergence test

take $|\,|$ $\Rightarrow \sum \left|\frac{\cos n}{n^2}\right| = \sum \frac{|\cos n|}{n^2}$

$|\cos n| \leq 1 \Rightarrow \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$

conv $P = 2 > 1$

$\sum \frac{1}{n^2}$ conv. using C.T $\Rightarrow \sum \frac{|\cos n|}{n^2}$ conv.

using ~~comparison~~ convergence test $\sum \frac{\cos n}{n^2}$ is conv.

No.

③ $\sum_{1}^{\infty} \frac{\sin(3n)}{n^4+1}$

using convergence test, take | |.

$$\sum \left| \frac{\sin 3n}{n^4+1} \right| = \sum \frac{|\sin 3n|}{n^4+1}$$

$$\frac{|\sin 3n|}{n^4+1} \leq \frac{1}{n^4+1} < \frac{1}{n^4}$$

$n^4+1 > n^4$

$\frac{1}{n^4+1} < \frac{1}{n^4}$

conv $\quad$ conv $\quad$ conv $p=4>1$

using comparison test

$\sum \frac{|\sin 3n|}{n^4+1}$ conv. $\xrightarrow{\text{using}}$ convergence test $\quad \sum \frac{\sin(3n)}{n^4+1}$ is conv.

Q: Is the following conv. or div.

① $\sum_{1}^{\infty} \ln\left(\frac{n}{4n+1}\right)$

using divergence test $\lim_{n\to\infty} \ln \frac{n}{4n+1} = \ln\left(\lim_{n\to\infty} \frac{n}{4n+1}\right)$

$= \ln \frac{1}{4} \neq 0$ div

② $\sum (-1)^n \cos\left(\frac{\pi}{n}\right)$

using div test $\lim_{n\to\infty} (-1)^n \cos \frac{\pi}{n} = \pm 1 \cdot \cos \frac{\pi}{\infty}$

$= \pm 1 \cdot \cos 0$

$= \pm 1 \cdot 1 = \pm 1 \neq 0$ div

Q: is the following ($\sum_{1}^{\infty} (-1)^n \frac{1}{n!}$) conv or div?

$* \quad |a_n| = \left| (-1)^n \frac{1}{n!} \right| = \frac{1}{n!}$

$|a_{n+1}| = \frac{1}{(n+1)!}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{(n+1)!} * \frac{n!}{1}$$

$$\lim \left(\frac{1}{(n+1)\,n!} * \frac{n!}{1} \right) = \frac{1}{\infty} = 0 < 1 \Longrightarrow \underline{\text{conv}}$$

<u>Def</u>: ① A series $\sum a_n$ is said to be Absolute convergent if $\sum |a_n|$ is convergent

② A series $\sum a_n$ is said to be conditionally convergent if $\sum |a_n|$ is ① div and $\sum a_n$ ② is conv

Q: which of the following is <u>Absolute</u> convergent or <u>conditionally</u> convergent :-

① $\sum_{1}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n$

$\sum \left(\frac{1}{2}\right)^n$ geom. $|r| = \frac{1}{2} \Rightarrow$ conv.

Absolutely convergent

② $\sum_{1}^{\infty} (-1)^n \cdot \frac{1}{n}$

taking $| \ |$, $\sum \frac{1}{n}$ div. $p=1$

but $\sum (-1)^n \cdot \frac{1}{n}$ conv.
by Alternating test
$\Rightarrow$ conditionally convergent

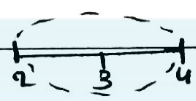
③ $\sum (-1)^n \cdot \frac{1}{n^2}$

taking $| \ |$, $\sum \frac{1}{n^2}$ conv. $p=2>1$ $\Rightarrow$ Absolutely convergent

* Radius of the interval :-

① $[2,4]$

rad $= \frac{4-2}{2} = 1$



② $(2,4)$

rad $= \frac{4-2}{2} = 1$

③ $(-\infty,\infty)$

$\frac{\infty-\infty}{2} = \frac{\infty}{2} = \infty$

④ $\{9\}$

Rad $= \frac{9-9}{2} = 0$

No.

* 11.8 : power series

$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$

is power series in power of x

$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \dots$

is power series in power of $(x-a)$

*** interval of convergence = those values of x that make the series convergent

Ex: find the interval of convergence and the radius:-

① $\sum_{1}^{\infty} \overset{a_n}{x^n}$ using Root tests

$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |x^n|^{1/n} = \boxed{|x| < 1}$

To find Interval of convergence use either Ratio or Root test

$|x| < 1 \implies -1 < x < 1$

At $-1 \implies \sum (-1)^n$ geom $|r| = |-1| = 1$ div

At $1 \implies \sum 1^n$ geom $|r| = 1$ div

Interval of conv. is $(-1, 1)$

Rad $= 1$

No.

② $\sum_{1}^{\infty} \frac{(x-3)^n}{n}$ a_n

using Ratio test $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

$|a_n| = \frac{|x-3|^n}{n}$, $|a_{n+1}| = \frac{|x-3|^{n+1}}{n+1}$

$\lim_{n\to\infty} \frac{|x-3|^{n+1}}{n+1} * \frac{n}{|x-3|^n}$

$\lim_{n\to\infty} \frac{|x-3|\,|x-3|^n}{n+1} * \frac{n}{|x-3|^n}$

$= |x-3| * 1 = |x-3| < 1$

$-1 < x-3 < 1 \Longrightarrow 2 < x < 4$

At 2 $\Longrightarrow \sum \frac{(2-3)^n}{n} = \sum \frac{(-1)^n}{n}$

conv. By Alternating test

At 4 $\Longrightarrow \sum \frac{(4-3)^n}{n} - \sum \frac{1}{n}$ div

$\Longrightarrow$ interval of convergence is $[2, 4)$

Rad = 1

No.

(3) $\sum_{1}^{\infty} n!\, x^n$ a_n

$|a_n| = n!\ |x|^n$

$|a_{n+1}| = (n+1)!\ |x|^{n+1}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!\ |x|^{n+1}}{n!\ |x|^n}$$

لجميع قيم (x) ماعدا الصفر تكون div.

$$= \lim_{n\to\infty} \frac{(2+1)\ 2!\ |x|^2\ |x|}{n!\ |x|^n} = |x| \lim_{n\to\infty} (n+1)$$

$$= \infty$$

$\forall x$ except $x=0$

At $0 \Longrightarrow \sum n! * 0 = \sum 0 = 0+0+\cdots$
$= 0$ conv.

$\Longrightarrow$ interval of convergence $\{0\}$

(4) $\sum_{1}^{\infty} \frac{(-1)^n\, x^{2n}}{2^{2n}\,(n!)^2}$ a_n

$$|a_n| = \frac{|x|^{2n}}{2^{2n}\, n! * n!}, \quad |a_{n+1}| = \frac{|x|^{2n+2}}{2^{2n+2} * (n+1)!\ n!} * \frac{2^{2n}\, n! \cdot n!}{|x|^{2n}}$$

$$\lim_{n\to\infty} \frac{|x|^2}{4(n+1)^2} = 0 < 1$$

interval of convergence is $(-\infty, \infty) = \mathbb{R}$

No.

* Taylor and Maclaurin series

Def: مهم

① $$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \cdots$$

it's called the Taylor series ≡ power series in power of $(x-a)$.

② $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)(x)^n}{n!} = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$$

** it's called Maclaurin series ≡ Taylor series when $a=0$, power series in power of x:

Ex: find the Maclaurine series for the following:-

① $f(x) = e^x$

$$\sum_{n=0} \frac{f^{(n)}(0)x^n}{n!} = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!}$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$ مهم

$f(x) = e^x \to f(0) = e^0 = 1$
$f'(x) = e^x \to f'(0) = 1$
$\vdots$
$f^{(n)}(x) = e^x \to f^{(n)}(0) = 1$

No.

(2) $f(x) = e^{-x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \longrightarrow e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = e^{-x}}$$ حفظ

(3) $f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

$$e^x - e^{-x} = \frac{2x}{1!} + 0 + \frac{2x^3}{3!} + 0 + \frac{2x^5}{5!} + \cdots$$

حفظ

$$\frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

(4) $$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

⑤ $f(x) = \sin x$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!} = f(0) + \frac{f'(0)\,x}{1!} + \frac{f''(0)\,x^2}{2!} + \frac{f'''(0)\,x^3}{3!} + \frac{f^{(4)}(0)\,x^4}{4!} + \cdots$$

$$= 0 + \frac{1x}{1!} + 0 + \frac{-1x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$$

Q: $\sum_{1}^{\infty} \left(\frac{1}{2}\right)^n + \frac{\ln n}{n^3}$ is it conv. or div?

$\sum \left(\frac{1}{2}\right)^n + \sum \frac{\ln n}{n^3}$

$\downarrow$

Geometric

$|r| = \frac{1}{2} < 1$

conv. + conv. $\equiv$ conv.

$\sum \frac{\ln n}{n^3}$ using L.C.T

$b_n = \frac{1}{n^2}$

conv $p = 2 > 1$

p - series

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^3} * \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{\ln n}{n} \quad \frac{\infty}{\infty}!$$

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0 \Rightarrow \text{conv}$$

No.

⑦ $f(x) = \dfrac{1}{1-x}$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!} = f(0) + \frac{f'(0)\, x}{1!} + \frac{f''(0)\, x^2}{2!} + \frac{f'''(0)\, x^3}{3!} + \frac{f^{(4)}(0)\, x^4}{4!} + \cdots$$

$$= 1 + \frac{1x}{\cancel{1!}} + \frac{\cancel{2!}\, x^2}{\cancel{2!}} + \frac{\cancel{3!}\, x^3}{\cancel{3!}} + \frac{\cancel{4!}\, x^4}{\cancel{4!}} + \cdots$$ فقط

$$= 1 + x + x^2 + x^3 + x^4 + \cdots = \boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$$

$f(x) = \dfrac{1}{1-x} = (1-x)^{-1} \longrightarrow f(0) = 1 = 0!$

$f'(x) = \dfrac{-1}{(1-x)^2} = -1(1-x)^{-2} * -1 = (1-x)^{-2} \longrightarrow f'(0) = 1 = 1!$

$f''(x) = 2.1(1-x)^{-3} \longrightarrow f''(0) = 2.1 = 2!$

$f'''(x) = 3.2.1\,(1-x)^{-4} \longrightarrow f'''(0) = 3.2.1 = 3!$

$f^4(x) = 4.3.2.1\,(1-x)^{-5} \longrightarrow f^4(0) = 4.3.2.1 = 4!$

$f^n(0) = n!$

⑧ $f(x) = \dfrac{1}{1+x}$

$\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$

$\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \displaystyle\sum_{n=0}^{\infty} (-x)^n$

$= \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$

No.

(9) $f(x) = \dfrac{1}{2+x}$

$$= \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2} * \frac{1}{(1+\frac{x}{2})}$$

$$= \frac{1}{2} * \frac{1}{(1-(\frac{-x}{2}))}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{2+x} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{-x}{2}\right)^n$$

(10) $\dfrac{1}{1+x^2}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x^2} = \frac{1}{1-(x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

(11) $\dfrac{x^4}{2+x} = x^4 * \dfrac{1}{2+x}$

$$\frac{1}{2+x} = \frac{1}{2(1-(\frac{-x}{2}))} = \frac{1}{2} * \frac{1}{1-(\frac{-x}{2})} = \frac{1}{2} * \sum_{n=0}^{\infty} \left(\frac{-x}{2}\right)^n$$

$$= \frac{1}{2} \sum_{n=0} \frac{(-1)^n x^n}{2^n}$$

$$= \sum_{n=0} \frac{(-1)^n x^n}{2^{n+1}}$$

$$\frac{x^4}{2+x} = x^4 * \sum_{n=0} \frac{(-1)^n x^n}{2^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+4}}{2^{n+1}}$$

(12) $\cos^2 x$

$$\cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \Longrightarrow \cos^2 x = \frac{1}{2} + \frac{1}{2}\sum_{n=0} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

(13) $$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\sum_{n=0} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

(14) $f(x) = (1+x)^K \longrightarrow f(0) = 1$

$f'(x) = K(1+x)^{K-1} \longrightarrow f'(0) = K$

$f''(x) = K(K-1)(1+x)^{K-2} \longrightarrow f''(0) = K(K-1)$

$f'''(x) = K(K-1)(K-2)(1+x)^{K-3} \longrightarrow f'''(0) = K(K-1)(K-2)(K-3$

$\Longrightarrow f^{(n)}(x) = K(K-1)(K-2)\ldots \qquad f^{(n)}(x)$?!!

$f^{(n)}(0) = K(K-1)(K-2)\;(K-(n+1))$

Maclaurin series is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!} = \sum_{n=0}^{\infty} \frac{K(K-1)(K-2)\cdots x^n (K-n+1}{n!}$

$(1+x)^K \quad \forall\, K$

No.

⑮ $f(x) = \frac{1}{\sqrt{1-x}} = \frac{1}{(1-1)^{1/2}} = (1-x)^{-1/2}$

$= (1 + (-x))$

** نضع بدل $(x) \leftarrow -x$

بدل $(k) \leftarrow \frac{-1}{2}$

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})(\frac{-1}{2}-1)(\frac{-1}{2}-2)\cdots(\frac{-1}{2}-n+1) * (-x)^n}{n!}$$

Note : لإيجاد (the sum of series) ننتبه لل (!) (Alternating)

- إذا كان المضروب $n!$ نفكر ب e^x او ب e^{-x}
- إذا كان المضروب $2n!$ نفكر ب cos او cosh
- إذا كان المضروب $(2n+1)!$ نفكر ب sin او sinh
- ننظر لـ (-1^n) موجودة في $\sin x, \cos x, e^{-x}$

Rules :

① $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

② $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

③ $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

No.

④ $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{2n!}$

⑤ $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

⑥ $\sinh x = \sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}$

⑦ $\cosh x = \sum_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!}$

Ex: find the sum

① $\dfrac{2}{1!} + \dfrac{4}{2!} + \dfrac{8}{3!} + \dfrac{16}{4!} + \cdots = \sum_{n=1}^{\infty} \dfrac{2^n}{n!}$

$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$

$\sum_{n=1}^{\infty} \dfrac{2^n}{n!} = \sum_{n=0}^{\infty} e \dfrac{2^n}{n!} - (n=0) = e^2 - \dfrac{2^0}{0!} = e^2 - 1$

② $1 - \ln 3 + \dfrac{(\ln 3)^2}{2!} - \dfrac{(\ln 3)^3}{3!} + \dfrac{(\ln 3)^4}{4} + \cdots$

$= \sum_{n=0} (-1)^n \dfrac{(\ln 3)^n}{n!}$, $e^{-x} = \sum \dfrac{(-1)^n x^n}{n!}$

$e^{-\ln 3} = e^{\ln 3^{-1}} = 3^{-1} = \dfrac{1}{3}$

No.

③ $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \dots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!}$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\boxed{\sin\pi = 0}$

Ex: find the taylor series for the following :-

① $f(x) = e^x$ at $a = 3$

$e^x = e^{x-3+3} = e^{x-3} \cdot e^3$

$= e^3 \cdot \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}$

$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$

② $f(x) = \sin x$ at $a = \frac{\pi}{2}$

$\sin x = \sin\left(\left(x - \frac{\pi}{2}\right) + \frac{\pi}{2}\right)$

$\sin\left(x - \frac{\pi}{2}\right)\cos\frac{\pi}{2} + \cos\left(x - \frac{\pi}{2}\right)\sin\frac{\pi}{2}$

$1 \cdot \cos\left(x - \frac{\pi}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x - \frac{\pi}{2}\right)^{2n}}{(2n)!}$

No.

③ $f(x) = e^x$ at $a = -3$

$e^x = e^{x+3-3} = e^{-3} \cdot e^{x+3}$

$= e^{-3} \sum_{n=0} \frac{(x+3)^n}{n!}$

** Differentiation and integration of power series:-

$\sum_{n=0}^{\infty} C_n (x-a)^n$ = power series in power of $(x-a)$

constant

① $\frac{d}{dx} \sum_{n=0}^{\infty} C_n (x-a)^n = \sum_{n=1}^{\infty} n C_n (x-a)^{n-1}$

② $\int \sum_{n=0}^{\infty} (C_n (x-a)^n) \, dx = \sum_{n=0}^{\infty} \frac{C_n (x-a)^{n+1}}{n+1} + C$

تكامل غير محدد

* Revision :- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

No. ______

Ex: find the power series for $\ln(1-x)$ using integration?

$$-\ln(1-x) = \int \frac{1}{1-x}\,dx$$

$$\ln(1-x) = -\int \frac{1}{1-x}\,dx$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \int \frac{1}{1-x}\,dx = \int \sum x^n$$

$$\int \frac{1}{1-x}\,dx = \sum_{n=0} \frac{x^{n+1}}{n+1} + C$$

$$\ln(1-x) = -\sum_{n=0} \frac{x^{n+1}}{n+1} + C$$

take $x=0 \longrightarrow \ln 1 = \sum 0 + C$

$$0 = 0 + C \rightarrow \boxed{C=0}$$

Ex: find the power series for $\tan^{-1}x = f(x)$?

Ex: find the power series for $\ln(1+x)$?

$$\ln(1+x) = \int \frac{1}{1+x}\,dx$$

$$= \int \left(\sum (-1)^n x^n\right) dx$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0} -x^n = \sum_{n=0} (-1)^n x^n$$

$$\boxed{\ln(1+x) = \sum_{n=0} (-1)^n \frac{x^{n+1}}{n+1} + C}$$

to find c tak $x=0$

$$\ln 1 = 0 = \sum 0 + C \rightarrow \boxed{C=0}$$

No.

Ex: find the power series for $\tan^{-1}x = f(x)$?

$$\tan^{-1}x = \int \frac{1}{1+x}\, dx$$

$$= \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx$$

$$\boxed{\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C}$$

$$\frac{1}{1-x} = \sum x^n$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$= \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0} (-1)^n x^{2n}$$

take $x=0 \Rightarrow \tan^{-1}0 = \sum (-1)^n \cdot 0 + C$

$$0 = 0 + C \longrightarrow \boxed{C=0}$$

Ex: use differentiation to find the power series of $f(x) = \frac{1}{(1-x)^2}$?

$$\frac{1}{(1-x)^2} = \frac{d}{dx} = \frac{1}{1-x} = \frac{d}{dx} \sum_{n=0}^{\infty} x^n$$

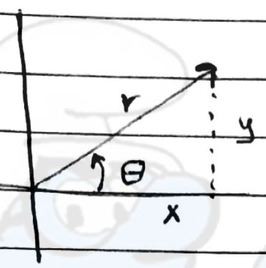
$$= \sum_{n=1}^{\infty} n x^{n-1}$$

No.

** polar coordinates **

→ Cartesian Coordinates (x, y)

→ polar Coordinates (r, θ)



$\cos\theta = \frac{x}{r}$

$\boxed{x = r\cos\theta}$

$\sin\theta = \frac{y}{r}$

$\boxed{y = r\sin\theta}$

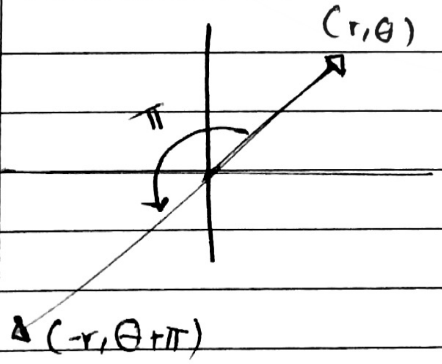
$\boxed{x = r\cos\theta}$ --- ①

$\boxed{y = r\sin\theta}$ --- ②

$\boxed{x^2 + y^2 = r^2}$ --- ③

Note: cartesian coordinate have unique representation but polar coordinate have infinite representation

$\tan\theta = \frac{y}{x} \Rightarrow \boxed{\theta = \tan^{-1}\frac{y}{x}}$ --- ④



No.

Ex: change $(2, \frac{\pi}{6})$ to cartesian coordinate?

$x = r\cos\theta = 2\cos \pi/6 = 2 * \frac{\pi}{3} = \sqrt{3}$

$y = r\sin\theta = 2\sin \pi/6 = 2 * 1/2 = 1$

$(x, y) \Rightarrow (\sqrt{3}, 1)$

Ex: change the following to polar coordinate?

① $(\sqrt{2}, \sqrt{2})$

$r^2 = x^2 + y^2 = (\sqrt{2})^2 + (\sqrt{2})^2 = 4$

$\tan\theta = \frac{y}{x} = \frac{\sqrt{2}}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$.

$(r, \theta) = (2, \frac{\pi}{4})$

مهم ② $(-\sqrt{2}, -\sqrt{2})$ ربع ثالث $(\pi + \theta)$

$x^2 + y^2 = r^2 = 4 \rightarrow \boxed{r = 2}$.

$\tan\theta = \frac{y}{x} = 1$

$\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

No.

م (3) $(-\sqrt{2}, \sqrt{2})$ الربع الثاني $(\pi - \theta)$

$x^2 + y^2 = r^2 = 4 \rightarrow \boxed{r = 2}$

$\tan\theta = \frac{y}{x} = -1$

$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$(r, \theta) = (2, \frac{3\pi}{4})$

م (4) $(\sqrt{2}, -\sqrt{2})$

$x^2 + y^2 = r^2 = 4 \longrightarrow \boxed{r = 2}$

$\tan\theta = -1$

$\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$(r, \theta) = (2, \frac{7\pi}{4})$

No.

change the equation into cartisian, what they represent ?

① $2 = r\cos\theta$

$2 = x \rightarrow$ line

$x = 2$

② $(r = 2\cos\theta)$ $* r$

$r^2 = 2r\cos\theta$

$x^2 + y^2 = 2x$

$x^2 - 2x + y^2 = 0 \longrightarrow (x-1)^2 + y^2 = 1$

completing square

circle with center $(1,0)$ and radius $\boxed{r = 1}$

③ $(r = 4\sin\theta)$ $* r$

$r^2 = 4r\sin\theta$

$x^2 + y^2 = 4y$

$x^2 + y^2 - 4y = 0$

$x^2 + (y^2) - 4y - 4 = 0$

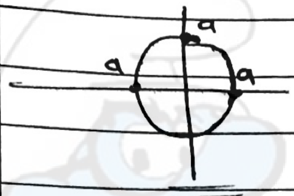
$x^2 + (y-2)^2 = 4 \longrightarrow$ circle with center $(0,2)$ and $\boxed{r = 2}$

Remark : the general form for the circle centered at (a,b) with radius r is $(x-a)^2 + (y-b)^2 = r^2$

No.

* polar curves

1 $r=a$ is $\boxed{\text{circl}}$ centered at $(0,0)$ with radius $=a$



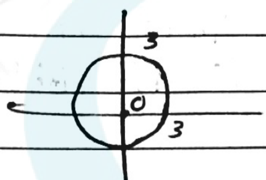
proof : $\boxed{r=a}$

$r^2=a^2$

$x^2+y^2=a^2$

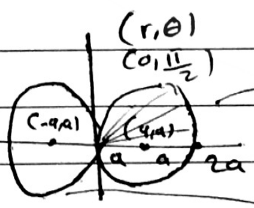
$\Rightarrow$ circle with center $(0,0)$ and $r=a$

ex: $r=3$



2 $r=2a\cos\theta$ $\boxed{\text{Circles}}$

$r=-2a\cos\theta$



* $r=2a\cos\theta$

$\Rightarrow$ circle centered at $(a,0)$ with radius $=a$

* $r=-2a\cos\theta$

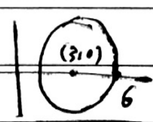
circle centered at $(-a,0)$ with radius a

ex: $r=6\cos\theta$

$6=2a \rightarrow a=3$

circle center $(3,0)$

$r=3$



No.

proof: $r = 2a\cos\theta \quad * r$

$r^2 = 2ar\cos\theta$

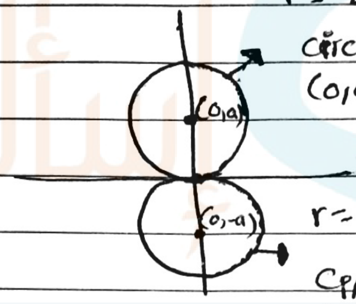
$x^2 + y^2 = 2ax \Rightarrow x^2 - 2ax + y^2 = 0$

$x^2 - 2ax + a^2 - a^2 + y^2 = 0$

$(x-a)^2 + y^2 = a^2$

$\boxed{3}$ $r = 2a\sin\theta$ للأعلى

$-$ $r = -2a\sin\theta$ للأسفل

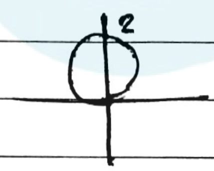


$r = 2a\sin\theta$
circle centered at $(0,a)$ with $\boxed{r=a}$

$r = -2a\sin\theta$
circle centered at $(0,-a)$ with $\boxed{r=a}$

Ex: $r = 2\sin\theta$

$\boxed{a=1} \rightarrow$ circle centered at $(0,1)$, $r=1$

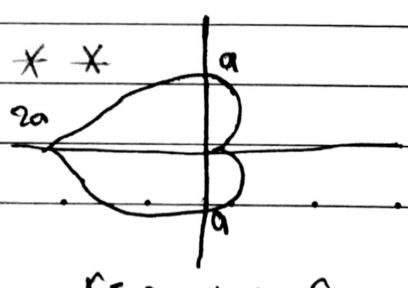


$\boxed{4}$ cardioid

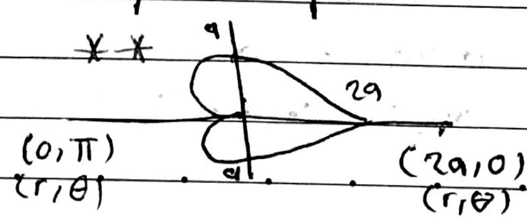
$r = a + a\cos\theta$

$r = a - a\cos\theta$

θ	0	$\pi/2$	π
r	2a	a	0



$r = a - a\cos\theta$

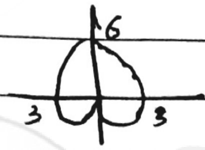


$r = a + a\cos\theta$

No.

Sketch: $r = 3(1 + \sin\theta)$

Sol'n: $r = 3 + 3\sin\theta \rightarrow$ cardioid



*poolar curves:-

① $\theta = \theta \rightarrow$ line

② $r = a \rightarrow$ circle with center $(0,0)$ and $\boxed{r = a}$

③ $*\, r = 2a\cos\theta$; $= -2a\cos\theta$ — circles but center Not $(0,0)$

$\cos\theta$

$*\, r = 2a\sin\theta$; $= -2a\sin\theta$

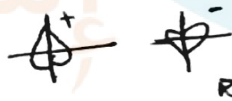
$\sin\theta$

④ cardioid

- $r = a \pm a\cos\theta$
- $r = a \pm a\sin\theta$

$\cos\theta$

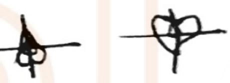
$\sin\theta$



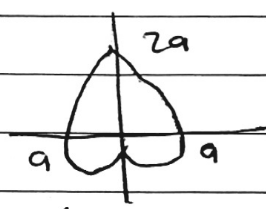
⑤ cardioid with inner loop

$r = a \pm b\cos\theta$

$r = a \pm b\sin\theta$



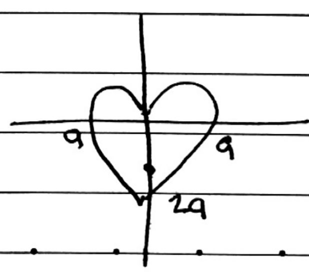
$*\quad r = a + a\sin\theta$



(r, θ)

$(0, \frac{3\pi}{2})$

$r = a - a\sin\theta$

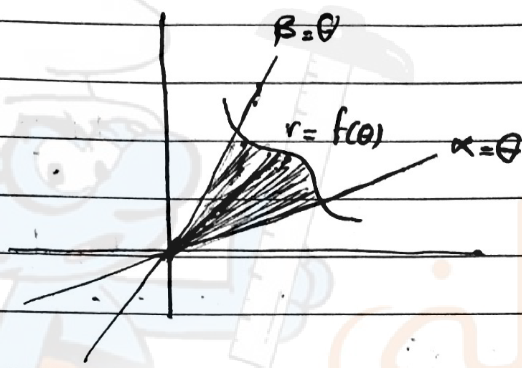


$(2a, \frac{3\pi}{2})$

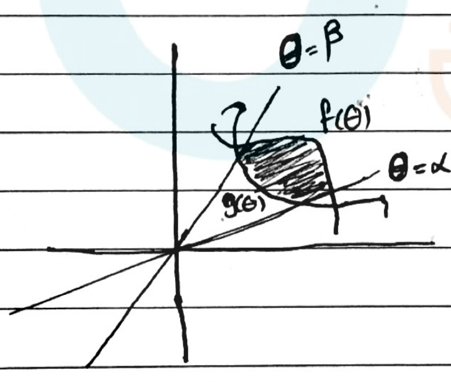
No.

* Area in polar coordinates :-

$$* \ A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2 \, d\theta$$

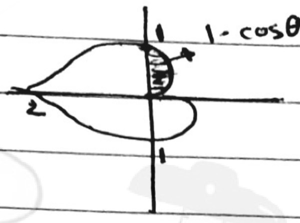


$$* \ A = \frac{1}{2}\int_{\alpha}^{\beta} \left(f^2(\theta) - g^2(\theta) \right) d\theta$$



No.

Ex: find the area of the region in the first quadrant within $\boxed{r = 1-\cos\theta}$?



$$A = \frac{1}{2}\int_0^{\pi/2} (1-\cos\theta)^2 \, d\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \frac{1}{2} + \frac{1}{2}\cos\theta$$

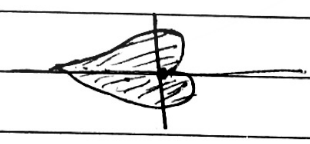
$$= \frac{1}{2}\int_0^{\pi/2} 1 - 2\cos\theta + \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta \, d\theta$$

$$= \frac{1}{2}\left(\frac{3\theta}{2} - 2\sin\theta + \frac{1}{2}\frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2}$$

Ex: find the $\boxed{\text{whole area}}$ of $r = 1-\cos\theta$?

$$A = \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2 \, d\theta$$



OR

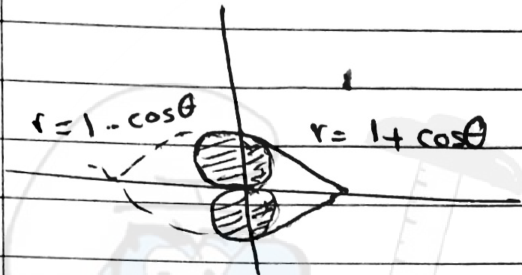
$$A = 2 * \frac{1}{2}\int_0^{\pi} (1-\cos\theta)^2 \, d\theta$$ (more easy)

↓

symmetry

No.

Ex: find the area enclosed by $r = 1 - \cos\theta$ and $r = 1 + \cos\theta$?



$r = r$ $\boxed{\text{intersection pts}}$

$1 - \cos\theta = 1 + \cos\theta$

$0 = 2\cos\theta$

$0 = \cos\theta$

$\cancel{\cos}\,\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{-\pi}{2}$

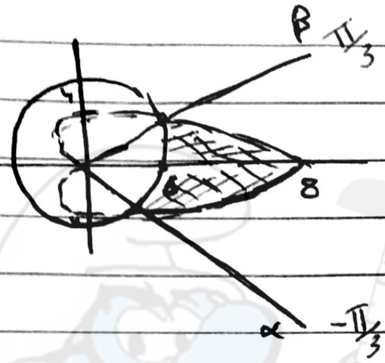
$$A = 4 \times \frac{1}{2} \int_0^{\pi/2} (1 - \cos\theta)^2 \, d\theta$$

<u>symmetry</u>

No.

Ex: find the area inside $\boxed{r = 4 + 4\cos\theta}$ and out side $r = 6$?!

so'ln :-



$r = r$ intersection pts

$6 = 4 + 4\cos\theta$

$2 = 4\cos\theta$

$\frac{1}{2} = \cos\theta \longrightarrow \theta = \frac{\pi}{3}$

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (4+4\cos\theta)^2 - 6^2 \, d\theta$$

more easy

or

$$A = 2 * \frac{1}{2}\int_{0}^{\pi/3} (4+4\cos\theta)^2 - 6^2 \, d\theta$$

No.

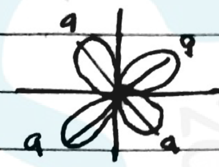
Roses :-

$r = a \sin n\theta$ $\qquad$ a: length of the petal

$r = a \cos n\theta$

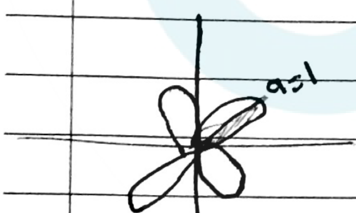
$n = 2 \longrightarrow$ عدد زوجي ⇐ عدد البتلات الضعف

$r = a\cos 2\theta$ ------ $r = a \sin 2\theta$



Ex: find the area enclosed by the rose $r = \sin 2\theta$?

$r = a$



$1 = \sin 2\theta$

$\frac{\pi}{2} = 2\theta$

$\frac{\pi}{4} = \theta$

لمقابلة

$A = 4 * \frac{1}{2} \int_0^{\pi/2} (\sin 2\theta)^2 \, d\theta$

4 ← عدد البتلات

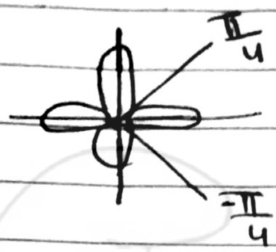
$A = 2 \int_0^{\pi/2} \frac{1 - \cos 4\theta}{2} \, d\theta$ $\qquad$ $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\sin^2 2\theta = \frac{1 - \cos 4\theta}{2}$

OR $A = 8 * 1/2 \int_0^{\pi/4} (\sin 2\theta)^2 \, d\theta$

No.

Ex: find the area enclosed by $\boxed{r = \cos 2\theta}$



$\boxed{r=0} \Longrightarrow 0 = \cos 2\theta$

$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

$\Longrightarrow A = 4 \times \frac{1}{2} \int_{-\pi/4}^{\pi/4} (\cos 2\theta)^2 \, d\theta$

OR $\quad A = 8 * \frac{1}{2} \int_{0}^{\pi/4} (\cos 2\theta)^2 \, d\theta \quad$ More easy

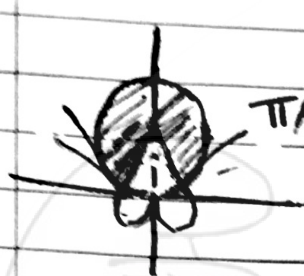
$= 4 \int_{0}^{\pi/4} \frac{1 + \cos 4\theta}{2} \, d\theta$

$= 2 \int_{0}^{\pi/4} 1 + \cos 4\theta \, d\theta$

$= 2 \left(1 + \frac{\sin 4\theta}{4} \right) \Big|_{0}^{\pi/4}$

No.

Ex: find the curves inside $\boxed{r = 3\sin\theta}$ and outside $\boxed{r = 1 + \sin\theta}$



* intersection points :-

$$3\sin\theta = 1 + \sin\theta$$

$$1 = 2\sin\theta$$

$$\sin\theta = \frac{1}{2}$$

$$\boxed{\theta = \frac{\pi}{6}}$$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2 - (1+\sin\theta)^2 \, d\theta$$

OR $\quad A = 2 \times \frac{1}{2}\int_{\pi/} (3\sin\theta)^2 - (1+\sin\theta)^2 \, d\theta$